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θ-Structures in Quantum Theory in View of Geometric Quantization

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Because the various topological effects (θ-structures) such as the Aharonov–Bohm effect, the anyon system, and non-Abelian statistics are pure quantum effects and should emerge naturally in a quantization procedure, we systematically discuss a general quantization scheme in a geometric formalism where wavefunctions are smooth sections of some vector bundles over configuration space. Following ideas of L. Schulman, M. Laidlaw, J. S. Dowker, and others, we choose those vector bundles to be the associated bundles of the universal covering space of configuration space. The θ-structures are shown to result from the fact that various vector bundles can be built over the universal covering space, which are labeled by the nonequivalent irreducible unitary representations of the fundamental group of configuration space. A flat connection description of θ-structures is also possible, owing to Milnor's theory.

1. INTRODUCTION

In the past several years, topological effects in quantum theory have been readdressed due to their possible relevance for understanding the fractional quantum Hall effect and high-T_c superconductivity (Wilczeck, 1990).

Many phenomena related to topology have been found in various fields, for example, Dirac's monopole (Dirac, 1958), the Aharonov–Bohm effect (Aharonov and Bohm, 1959), the chiral anomaly (Witten, 1983), solitons, instantons, the θ-vacuum (Witten, 1979), and more recently, anyon systems (Wen *et al.*, 1989). Different approaches to understanding these topological effects in quantum theory have been proposed. Roughly speaking, they may be divided into two classes, model-based approaches and quantization-based approaches. The former, such as Wilczeck's (1982) model for anyon gases,

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has the advantage of concreteness and clarity of the physical picture. However, it seems to the authors that the latter approach is more acceptable. There are two reasons supporting this conclusion: (1) Not all existing topological effects can be assigned a meaningful "model." (2) Since those topological effects appear only in quantum theory, it is natural to suppose that they emerge at the stage of the quantization procedure from classical theory to the corresponding quantum theory.

Investigations of topological effects in quantum theory along this approach started around the 1970s by Schulman, Laidlaw, DeWitt-Morette, Dowker, and others. They argued that, when performing path integral quantization in the multiconnected configuration space Q, the propagator should generally take the form (Schulman, 1968)

$$K(q', t'; q, t) = \sum_{\alpha} \chi(\alpha) K^{\alpha}(q', t'; q, t) \tag{1.1}$$

where K^{α} is a kernel summing over paths that belong to the same homotopic class $\alpha \in \pi_1(Q)$. It was proved that the weight factors $\{\chi(\alpha)\}$ must form a scalar unitary representation of the fundamental group $\pi_1(Q)$ to fit general requirements on propagators (Laidlaw and DeWitt, 1971).

Dowker (1972) rederived this result by performing the usual path integration over the universal covering space $\tilde{Q}$ of Q, with an argument that wavefunctions single-valued in $\tilde{Q}$ could be multivalued in Q.

Further generalization of Dowker's idea is straightforward. That is, "wavefunctions" of a quantum theory could be multicomponent, multivalued functions in Q (but single-valued in $\tilde{Q}$) e.g., $\tilde{\Psi} \in C^{\infty}(\tilde{Q}, \mathscr{C}^N)$. The physical interpretation of the multicomponentness of "wavefunctions" is related to "internal freedoms" of quantum systems as usual. Under this general situation, a similar result analogous to (1.1) can be readily found (Sudarshan *et al.*, 1988)

$$\tilde{\Psi}_n(\gamma \tilde{q}) = \sum_{m=1}^{N} U_{nm}(\gamma) \tilde{\Psi}_m(\tilde{q}) \tag{1.2}$$

where $U(\gamma)$ should form an N-dimensional unitary representation of $\pi_1(Q)$.

Obviously, when $\pi_1(Q)$ has various nonequivalent irreducible unitary representations (NIUP), each of them will bear a distinct quantum theory. Therefore, there is an *ambiguity* in quantizing a classical system with nontrivial configuration space. This kind of ambiguity is totally different from dynamical ones such as the ordering ambiguity in conventional quantum mechanics (Bao and Zhu, 1991). This kind of ambiguity plays a key rule in understanding various θ-structures in quantum theory. To be concrete, let us give some well-known examples:

1. Aharonov–Bohm effect (Aharonov and Bohm, 1959):

$$\pi_1(Q)=\pi_1(\mathscr{R}^3-\mathscr{R})=\mathscr{Z}$$
$$\mathrm{Hom}(\pi_1(Q),\ U(1))=U(1) \tag{1.3}$$
$$\mathrm{Hom}(\pi_1(Q),\ U(N))=0, \qquad N>1$$

Therefore, the NIURs of the $\pi_1(Q)$ are all one-dimensional and can be labeled by a parameter θ. Adoption of different θ values means different quantizations and will lead to different quantum theories. (In the usual model for the Aharonov–Bohm effect, $\theta=q\Phi$, where q is the charge possessed by the particle going around a solenoid with magnetic flux Φ. The ambiguity here could be regarded as the ambiguity in choosing models with parameters q, Φ.)

2. Identical-particle system. The configuration space of n identical particles in $\mathscr{R}^d$ is (Laidlaw and DeWitt, 1971)

$$Q=(\mathscr{R}^{nd}-D^n)/S_n \tag{1.4}$$

where D^n is the configuration that any two particles occupy the same point in $\mathscr{R}^d$, and S_n is the permutation group of order n.

The fundamental group of the Q is given as

$$\begin{aligned}\pi_1(Q)&=S_n, \qquad d>2\\ \pi_1(Q)&=B_n, \qquad d=2\end{aligned} \tag{1.5}$$

where S_n is the permutation group and B_n is Artin's braiding group.

If only scalar quantum mechanics is concerned, since

$$\begin{aligned}\mathrm{Hom}(\pi_1(Q),\ U(1))&=Z_2, \qquad d>2\\ \mathrm{Hom}(\pi_1(Q),\ U(1))&=Z, \qquad d=2\end{aligned} \tag{1.6}$$

then a well-known conclusion can be drawn, in the spirit of (1.1), that only fermion and boson statistics exist when $d>2$, but fractional statistics (or more generally, θ-statistics) exists when $d=2$ (Sudarshan *et al.*, 1988).

However, as $\pi_1(Q)$ is non-Abelian, it admits high-dimensional NIUR. Therefore, in the spirit of (1.2), there exists *non-Abelian statistics*. [For historical reasons, the non-Abelian statistics for $d>2$, which is classified by $\mathrm{Hom}(S_n,\ U(N))$, commonly called *parastatistics*, while that for $d=2$, which is classified by $\mathrm{Hom}(B_n,\ U(N))$, is commonly called as *exotic statistics* (Date *et al.*, 1990).]

Now, we have seen that the proposition (1.1) or more generally (1.2) indeed has a key role in the understanding of various θ-structures in quantum theory via quantization.

However, (1.1) and (1.2) are proposed *a priori*. It is crucial to understand them in a general background concerning quantization for both mathematical interest and practical applications. This is actually the main purpose of this paper. As θ-structures are related to global properties, there is a good reason to treat them in light of geometric quantization, which is essentially the "globalization" of canonical quantization, as the title of this paper suggests.

The organization of this paper is as follows. In Section 2, we discuss the general formalism of quantization based on the geometric viewpoint where the Hilbert spaces upon which quantum theory is based are established as the spaces of square-integrable smooth sections of some vector bundles over configuration space Q. (For completeness, a brief discussion on dynamical problems is also presented.) In Section 3, we discuss how the vector bundles can be found through both mathematical techniques and physical intuition. In Section 4, we derive (1.2) as a natural result of our quantization procedure and relate it to a concrete concept, flat connections on bundles, which seems to be more applicable for practical purposes. Section 5 presents a summary of our results and an extended discussion.

2. GENERAL FORMALISM

In this section, we present a general formalism of quantization from a geometric viewpoint. The discussion is based on the Hilbert space approach to quantum mechanics, where a quantum pure state is related to a ray in some Hilbert space $\mathscr{H}$ and a quantum observable is related to a self-adjoint operator acting on $\mathscr{H}$.

It is important that the above statements should be properly understood. For example, not every vector in $\mathscr{H}$ can be regarded as a physically realizable state (e.g., when there is a superselection rule), and not every classical observable can be associated with a self-adjoint operator on $\mathscr{H}$ without disobeying other physical requirements (Isham, 1984).

Now, insofar as "quantization" is concerned, given a specific classical system, (1) how do we construct the Hilbert space? (2) how do we select the subset of classical observables that can be "quantized?" and (3) how do we construct the concrete self-adjoint operators that are to be associated with them?

According to the ideas of geometric quantization and those initiated by Schulman, DeWitt, Dowker, and others as mentioned in Section 1, it is quite reasonable to construct the Hilbert space $\mathscr{H}$ from the space of smooth sections of some vector bundle E over the classical phase space M. (Needless to say, cross sections of a vector bundle have a natural vector space structure.) It is well known in geometric quantization that in order to be in accord with the uncertainty principle, such a constructed $\mathscr{H}$ is too large and must be restricted via the mechanism of "polarization" (Woodhouse, 1980). For the purpose of this paper, it is helpful not to deal with the most general situations, but to limit our considerations to those classical systems whose phase space M is a cotangent bundle of configuration space Q, i.e., $M = T^*Q$. The Schrödinger polarization will be adopted, whose results are most comparable with the conventional language of quantum mechanics (Sniatycki, 1980). As a result, the Hilbert space H now can be equally regarded as being constructed from $\Gamma(E)$, the space of smooth sections of some vector bundle E over classical configuration space Q.

In order for those sections to indeed form a Hilbert space, it is required that the vector bundle E should be a Hermitian vector bundle. Namely, there is an assignment of an inner product $\langle *, * \rangle_q$ to each vector space fiber $\mathscr{C}_q^n$ such that, if Ψ_1, Ψ_2 are two smooth sections of E, then the map $q \mapsto \langle \Psi_1(q), \Psi_2(q) \rangle_q$ is smooth. If such a Hermitian structure exists in E, then the inner product of two smooth sections $\Psi_1, \Psi_2 \in \Gamma(E)$ can be defined as

$$\langle \Psi_1, \Psi_2 \rangle = \int_Q \langle \Psi_1(q), \Psi_2(q) \rangle_q \, d\mu(q) \tag{2.1}$$

Therefore, the Hilbert space $\mathscr{H}$ upon which quantum mechanics is based can be chosen as $\mathscr{H} = L^2(\Gamma(E); d\mu)$, where L^2 means that the smooth sections chosen are square-integrable in the sense of (2.1).

However, question 1 has not been completely answered. As there are many vector bundles that can be constructed from the same configuration space Q (for example, vector bundles with different fiber dimensions or vector bundles with identical fiber dimension but belonging to different Chern classes), how can we choose a preferred one from those possible constructed vector bundles? Or, if there is no preference, what are the differences among resulting quantum theories built upon different vector bundles?

At this stage, we only point out that the answer to these questions are closely related to the θ-structures in quantum theories as mentioned in Section 1. Detailed discussion is presented in the next section.

A complete formalism for quantization should include answers to questions 2 and 3, which are absolutely nontrivial and important. However, for the purpose of this paper (concerning θ-structures), they are not of primary interest and we will only mention briefly some general ideas concerning these dynamical questions.

The subset of classical observables to be quantized can be properly selected through group theory considerations. The procedure mainly goes as follows (Isham, 1984).

Step 1. Look for a finite-dimensional Lie group G of symplectic transformations of phase space M. The Lie group G must be chosen so that each one-parameter subgroup of G generates a one-parameter family of symplectic transformations with a corresponding Hamiltonian vector field on M. The set of these fields is denoted h.

Step 2. Try to associate an observable f with each Hamiltonian vector field X in h (i.e., $X=\xi_f$) in such a way that the Poisson brackets of the corresponding set of observables close to give the Lie algebra $L(G)$ again.

Step 3. If the above steps are successful, one may identify the subset of classical observables to be quantized with those chosen.

After selecting the subset of classical observables to be quantized, the subsequent work of quantization is essentially to study the irreducible unitary representations of the "canonical group" G on some Hilbert space $\mathscr{H}=L^2(\Gamma(E), d\mu)$ that has been determined previously.

The unitary representation of G on this $\mathscr{H}$ can be formally defined as follows (Wallach, 1973):

$$(U(g)\Psi)(x)=\left(\frac{d\mu_g}{d\mu}(x)\right)^{1/2} l_g^\dagger \Psi(g^{-1}x) \tag{2.2}$$

where the Radon–Nikodyn derivative $d\mu_g/d\mu$ is desired in order to guarantee that the representation is unitary, and a lift-action $l_g^\dagger: \mathscr{C}_x^n \to \mathscr{C}_{gx}^n$ is introduced so that the left side and right side of (2.2) are defined on the same fiber of E. It can be readily shown that, in order that (2.2) indeed defines a unitary representation of G, the lift-action $l_g^\dagger$ should obey the following properties:

$$\begin{gathered} l_{g_1}^\dagger \circ l_{g_2}^\dagger = l_{g_1 g_2}^\dagger \\ \langle l_g^\dagger u, l_g^\dagger v\rangle_{gx} = \langle u, v\rangle_x \end{gathered} \tag{2.3}$$

Now, the question is whether such a lift-action exists. If it exists, how many nonequivalent lifts are there? These problems, commonly referred to as the "G-lift problem in bundles" (Gottlieb, 1978), are important since non-equivalent lifts will lead to different representations of G and thus to different quantum theories. A complete solution of the G-lift problem for general situations awaits further, mathematical investigation. In the physical sense, we realize that the G-lift problem involves a kind of ambiguity in quantization, which is more or less comparable with the "ordering ambiguity" in the canonical quantization of conventional quantum mechanics (Bao and Zhu, 1991).

As the purpose of this paper concerns understanding θ-structures in quantum theory, which does not include the G-lift problem, we will discuss this aspect of quantization elsewhere and devote the next section to the problem of constructing vector bundles E over configuration space Q, which is directly related to θ-structures, as indicated previously.

3. VECTOR BUNDLES OVER CONFIGURATION SPACE

As mentioned in the last section, when quantizing a given classical system with phase space $M = T^*Q$, the Hilbert space upon which quantum theory is based is generally chosen as $\mathscr{H} = L^2(\Gamma(E), d\mu)$, where E is some Hermitian vector bundle over Q, i.e., $\mathscr{C}^n \to E \to Q$.

Now the problem is how various vector bundles can be constructed over Q. Different vector bundles usually lead to different Hilbert spaces and thus to different quantum mechanics corresponding to the same classical model.

According to general fiber bundle theory, there is a standard method of constructing Hermitian vector bundles over some space Q. That is, we first look for some "master" principal K-bundle P over Q, i.e., $K \to P \to Q$, and if $k \mapsto \mathscr{U}(k)$ is a unitary representation of the Lie group K on $\mathscr{C}^n$, then an associated Hermitian vector bundle, denoted as $P \otimes_{\mathscr{U}} \mathscr{C}^n$, can be constructed as follows (Eguchi *et al.*, 1980):

$$\mathscr{C}^n \to P \underset{\mathscr{U}}{\otimes} \mathscr{C}^n \to Q, \qquad \text{where} \quad (p, v) = (pk, \mathscr{U}(k^{-1})v) \tag{3.1}$$

The Hermitian structure on this vector bundle is defined as

$$\langle [p, u], [pk, v] \rangle_q = \langle u, \mathscr{U}(k)v \rangle_{\mathscr{C}^n} \tag{3.2}$$

where the right-hand side is natural inner product on $\mathscr{C}^n$.

This vector bundle can also be obtained through a two-stage construction as follows:

Given a "master" principal K-bundle P over Q, if $k \to \mathscr{U}(k)$ is a unitary representation of K on $\mathscr{C}^n$, then $\mathscr{U} \in \mathrm{Hom}(K, U(n))$ and we can first construct the principal $U(n)$ bundle,

$$U(n) \to P \underset{\mathscr{U}}{\otimes} U(n) \to Q, \qquad \text{where} \quad (p, U) = (pk, \mathscr{U}(k^{-1})U) \tag{3.3}$$

Then, since $U(n)$ acts on $\mathscr{C}^n$, we can further form the associated vector bundle,

$$\mathscr{C}^n \to \left(P \underset{\mathscr{U}}{\otimes} U(n)\right) \underset{\mathscr{U}(n)}{\otimes} \mathscr{C}^n \to Q, \qquad \text{where} \quad ([p, u], v) = ([p, U]U', U'^{-1}v) \tag{3.4}$$

As there is a vector bundle isomorphism defined by

$$\begin{gathered}\left(P \underset{\mathscr{U}}{\otimes} U(n)\right) \underset{\mathscr{U}(n)}{\otimes} \mathscr{C}^n \to P \underset{\mathscr{U}}{\otimes} \mathscr{C}^n \\ ([p, U], v) \mapsto [p, Uv]\end{gathered} \tag{3.5}$$

the two vector bundles defined by (3.1) and (3.4) are actually isomorphic.

Obviously, these vector bundles $P \otimes_{\mathscr{U}} \mathscr{C}^n$, constructed in a standard way from the master principal bundle $K \to P \to Q$, are classified by the non-equivalent irreducible unitary representations (NIUR) of the Lie group K.

Now our problem concerning the construction of the Hilbert space is reduced to what "master" principal bundle should be adopted. A purely mathematical consideration of constructing vector bundles will hardly answer this problem. However, in no sense can quantization be regarded as a purely mathematical exercise. The physical ideas behind it are of essential importance. In fact, as indicated in the Introduction, if Feynman's idea of "summing over history" is accepted, there is a strong physical intuition, as suggested by Schulman, DeWitt, Dowker, and others, that the universal covering space $\tilde{Q}$ of configuration space Q, which itself is a principal bundle $\pi_1(Q) \to P \to Q$, should be taken as our "master" principal bundle for quantization.

Therefore, with the combined use of physical intuition and mathematical techniques, we have obtained a family of Hilbert spaces from a single classical configuration space Q as follows:

$$\mathscr{H}_{\mathscr{U}(\pi_1(Q))} = L^2\left(\Gamma\left(\tilde{Q} \underset{\mathscr{U}}{\otimes} \mathscr{C}^n, d\mu\right)\right) \tag{3.6}$$

This family of Hilbert spaces is obviously labeled by the NIUR of $\pi_1(Q)$.

In the next section, we will see how most θ-structures in quantum theory can be well understood in this formalism.

4. θ-STRUCTURES AND FLAT CONNECTIONS

Through a careful investigation of general quantization, we have constructed a family of Hilbert spaces $\mathscr{H}_{\mathscr{U}(\pi_1(Q))}$ as in (3.6) from a given classical system. In this section, as the main purpose of this paper, we discuss the physical meaning of our construction and relate this construction to various θ-structures in quantum theory.

As mentioned in the Introduction, the θ-structures in quantum theory can be best described by (1.1) or (1.2) which are proposed *a priori*. Now, we will try to derive them with the aid of the above-discussed quantization scheme.

When viewing Ψ as a smooth section of a vector bundle E, it is easy to see that (1.2) has something to do with parallel transport between fibers of E, and therefore the concept of connection on a bundle will play a role here. Actually, it seems that it is the flat part of the connection on a bundle that has to do with θ-structures (Yu *et al.*, 1991). Therefore, we first discuss connections on the bundle,

$$\mathscr{C}^n \rightarrow \tilde{Q} \underset{\mathscr{U}}{\otimes} \mathscr{C}^n \rightarrow Q \tag{4.1}$$

An important result closely related to our consideration has been given by Milnor (1975), which states that there is a natural flat connection on each of the $\tilde{Q} \otimes_{\mathscr{U}} U(n)$, and furthermore, any $U(n)$ bundle on Q with a flat connection is of the form $\tilde{Q} \otimes_{\mathscr{U}} U(n)$ for some unitary representation $\mathscr{U}$ of $\pi_1(Q)$.

Now, as there exists a vector bundle isomorphism defined by (3.5), we may equally well concentrate on the $\tilde{Q} \otimes_{\mathscr{U}} U(n)$ principal bundle to understand the relevant results of our general quantization. [Similar to scalar geometric quantization, where it is the $U(1)$ principal bundle rather than the complex line bundle that is considered.]

In order for global cross sections to exist, the principal bundle $\tilde{Q} \otimes_{\mathscr{U}} U(n)$ must necessarily be trivializable, i.e., there is a bundle morphism as follows:

$$\begin{gathered} \tilde{Q} \underset{\mathscr{U}}{\otimes} U(n) \rightarrow Q \otimes U(n) \\ [y, U] \mapsto (r(y), D(y)U) \end{gathered} \tag{4.2}$$

where $D(y\gamma) = D(y)\mathscr{U}(\gamma)$, $\forall \gamma \in \pi_1(Q)$. According to Milnor (1975), the map function $D: \tilde{Q} \rightarrow U(n)$ can be expressed via the natural flat connection on $\tilde{Q} \otimes_{\mathscr{U}} U(n)$, with boundary condition $D(y_0) = I$, as

$$D(y) = \mathscr{P} \exp\left(i \int_C A\right) \tag{4.3}$$

where C is any curve in Q whose lift to $\tilde{Q}$ joins y_0 to the point y [there is a curve lifting lemma (Dieudonne, 1972) that guarantees that such a lift exists and is unique once y_0 is specified]. A is defined to be $A = i\sigma^*\omega$, where ω is the natural flat connection on $\tilde{Q} \otimes_{\mathscr{U}} U(n)$ and $\sigma^*\omega$ is its pullback to $Q \otimes U(n)$.

Now let C_λ be a closed curve in Q which belongs to $\gamma \in \pi_1(Q)$. Then the condition $D(y\gamma) = D(y)\mathscr{U}(\gamma)$ and (4.3) imply that the unitary representation of γ can be equally expressed by the flat connection A as follows:

$$\mathscr{U}(\gamma) = \mathscr{P} \exp\left(i \oint_{C_\gamma} A\right) \tag{4.4}$$

This is a most important result that is relevant to the physical understanding of θ-structures.

As in (3.6), wavefunctions of quantum theory are smooth sections of $\tilde{Q} \otimes_{\mathscr{U}} \mathscr{C}^n$. By the general theory of fiber bundles, the set of smooth sections of the associated bundle $P \otimes_K F$ can be essentially defined (Chern, 1967)

$$\Gamma\left(P \underset{K}{\otimes} F\right) = \{\Psi \in C^\infty(P, F) \mid \Psi(pk) = k^{-1}\Psi(p)\} \tag{4.5}$$

Therefore, wavefunctions which now belong to $\Gamma(\tilde{Q} \otimes_{\mathscr{U}} \mathscr{C}^n)$ have the following property

$$\Psi(y\gamma) = \mathscr{U}(\gamma^{-1})\Psi(y), \qquad \Psi \in C^\infty(\tilde{Q}, \mathscr{C}^n) \tag{4.6}$$

Referring to (4.4), this can be equivalently expressed as

$$\Psi(y\gamma) = \left(\mathscr{P} \exp\left(-i \oint_{C_\gamma} A\right)\right)\Psi(y) \tag{4.7}$$

Therefore, we have rederived (1.2) as in (4.6) and, furthermore, the flat connection on the bundle has been included naturally in (4.7).

It is interesting to note that if our Hilbert space had not been chosen to be $L^2(\Gamma(\tilde{Q} \otimes_{\mathscr{U}} \mathscr{C}^n), d\mu)$, then (4.6) and (4.7) would generally be non-equivalent when regarded as the starting points of further considerations, since (4.6) is related to $\mathrm{Hom}(\pi_1(Q), U(n))$, while (4.7) is related to the holonomy group over bundles. However, as our Hilbert space *has been chosen* to be $L^2(\Gamma(\tilde{Q} \otimes_{\mathscr{U}} \mathscr{C}^n)$ for reasons mentioned in the last section, one may equally regard (4.6) or (4.7) as the starting point of further considerations, because of Milnor's theory.

5. CONCLUSION AND DISCUSSION

Recently, a possible "anyon explanation" for high-T_c superconductivity has given rise to new interest in θ-structures in quantum theory. With the argument that θ-structures are pure quantum effects and thus should emerge naturally in the procedure of quantization, we have systematically discussed a general quantization scheme in the geometric formalism where wavefunctions are smooth sections of some vector bundle over configuration space. (In other words, we considered the phase space T^*Q with Schrödinger polarization.) Our construction of the vector bundles is based on the standard construction of vector bundles from a "master" principal bundle chosen to be the universal covering space $\tilde{Q}$ of the configuration space Q. The physical idea behind our construction follows essentially notions initiated by Schulman, DeWitt, Laidlaw, Dowker, and others. The θ-structures naturally arise in our quantization procedure due to the variety of vector bundles that can be built from the master principal bundle $\tilde{Q}$. A flat connection description of θ-structures is also possible in a natural way, owing to Milnor's theory.

However, in spite of all its successes, our construction should not be regarded as a unique one or the most general one. Actually, the construction here seems to have the disadvantage that, if one regards wavefunctions that are smooth sections of vector bundles with fiber dimension greater than one, such as can describe "internal freedoms" of a corresponding quantum system, then the possible "internal freedoms" of the quantum systems quantized from a given classical system depend upon the fundamental group of configuration space.

A natural way to overcome this problem is to investigate *all* possible vector bundles that can be built up over a specific classical configuration space and take the Hilbert space to be the set of square-integrable smooth sections of these bundles. However, this would be not only of purely mathematical interest, but too ambitious as well. An alternative way, which seems to us to be more acceptable, is to regard the causes of the nontrivial topology of configuration space as some kind of restriction (or equivalently, as some kind of "gauge invariance" possessed by the system under consideration), and "quantize" the system first without considering the restrictions but putting in the requirement of "gauge invariance" afterward to limit the size of the Hilbert space to be "physical." In the quantization of Yang–Mills field theory, the BRST method is actually along these lines (Felder, 1989), and so is the Chern–Simons model construction for anyon systems (Frolich and Marchetti, 1991). It seems that this approach to quantization can be applied to more general situations (Sorkin, 1986), and we are investigating this.

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REFERENCES

Aharonov, Y., and Bohm, D. (1959). *Physical Review*, **115**, 485.

Bao, D., and Zhu, Z. Y. (1991). *Communications in Theoretical Physics*, **16**, 201.

Chern, S. S. (1967). *Complex Manifolds without Potential Theory*, Van Nostrand.

Date, G., Govindarajan, T., Sankavan, P., and Shankar, R. (1990). *Communications in Mathematical Physics*, **132**, 293.

Dieudonne, J. (1972). *Treatise on Analysis*, Vol. III, Academic Press, New York.

Dirac, P. A. M. (1958). *The Principles of Quantum Mechanics*, 4th ed., Oxford University Press, Oxford.

Dowker, J. S. (1972). *Journal of Physics A*, **5**, 936.

Eguchi, T., Gilkey, P., and Hanson, A. J. (1980). *Physics Reports*, **66**, 213.

Felder, G. (1989). *Nuclear Physics B*, **317**, 215.

Frohlich, J., and Marchetti, P. A. (1991). *Nuclear Physics B*, **356**, 533.

Gottlieb, D. H. (1978). In *Geometric Applications of Homotopy Theory*, Springer, Berlin.

Isham, C. J. (1984). In *Relativity, Groups and Topology II* (B. S. DeWitt and R. Stora, eds.), New York.

Laidlaw, M., and DeWitt, C. M. (1971). *Physical Review D*, **3**, 1375.

Milnor, J. (1975). *Commentarii Mathematici Helvetici*, **32**, 215.

Schulman, L. (1968). *Physical Review*, **176**, 1558.

Sniatycki, J. (1980). *Geometric Quantization and Quantum Mechanics*, Springer-Verlag, New York.

Sorkin, R. D. (1986). In *Topological Properties and Global Structure of Space-Time* (P. G. Bergmann and V. de Sabbata, eds.), Plenum Press, New York.

Sudarshan, E., Imbo, T., and Govindarajan, T. (1988). *Physics Letters B*, **213**, 471.

Wallach, N. R. (1973). *Harmonic Analysis on Homogeneous Spaces*, Marcel Dekker, New York.

Wen, X. G., Wilczeck, F., and Zee, A. (1989). *Physical Review B*, **39**, 11413.

Wilczeck, F. (1982). *Physical Review Letters*, **49**, 957.

Wilczeck, F. (1990). *Fractional Statistics and Anyonic Superconductivity*, World Scientific, Singapore.

Witten, E. (1979). *Physics Letters*, **86B**, 283.

Witten, E. (1983). *Nuclear Physics B*, **223**, 422.

Woodhouse, N. (1980). *Geometric Quantization*, Oxford University Press, Oxford.

Yu, Y., Zhu, Z. Y., and Lee, H. C. (1991). *Communications in Theoretical Physics*, **15**, 339.